Chapter4

Applications of Derivatives

Chapter4: Applications of Derivatives

In the previous chapter we focused almost exclusively on the computation of derivatives. In this chapter will focus on applications of derivatives. It is important to always remember that we didn't spend a whole chapter talking about computing derivatives just to be talking about them. There are many very important applications to derivatives.

The two main applications that we'll be looking at in this chapter are using derivatives to

determine information about graphs of functions and optimization problems. These will not be

the only applications however. We will be revisiting limits and taking a look at an application of derivatives that will allow us to compute limits that we haven't been able to compute previously.

(I)Tangent and Normal Lines

The derivative of a function at a point is the slope of the tangent line at this point. The **normal line** is defined as the line that is perpendicular to the tangent line at the point of tangency. Therefore, we have:

1) The equation of the tangent line for the curve $y = f(x)$ at the point (x_1, y_1) *is given by*

$$\frac{y - y_1}{x - x_1} = m, \quad \text{where } m = f'(x_1) := f'(x)\big|_{x = x_1}$$

2) The equation of the normal line for the curve $y = f(x)$ at the point $x = a$ *is given by*

$$\frac{y - y_1}{x - x_1} = -\frac{1}{m}$$

Ex $f(x)=\sqrt{x^2+3}$ the equation of the tangent and normal lines to the graph of at the point **(−1,2).**

Solution:

$$f(x)=(x^2+3)^{1/2}$$

$$f'(x)=\frac{1}{2}(x^2+3)^{-1/2}\cdot(2x)$$

$$f'(x)=\frac{x}{\sqrt{x^2+3}}$$

At the point $(-1,2)$, $f'(-1)=-½$ and the equation of the line is

$$y-y_1=m(x-x_1)$$

$$y-2=-\frac{1}{2}(x+1)$$

$$2y-4=-x-1$$

$$x+2y=3$$

Similarly, the equation of the normal line at the point (−1,2) is

$$\frac{y-2}{x+1}=2$$

$$y-2=2(x+1)$$

$$y-2x-4=0$$

Example: Find the equation of the tangent and normal lines to the graph of $y=x^2$ at the point **(−1,1).**

Solution: By differentiating, we have

$$\because f(x)=x^2$$

$$\Rightarrow f(x)=2x$$

$$\Rightarrow m=f'(-1)=-2$$

Hence, the equation of the tangent line at the point (−1,1) is

$$y-1=-2(x+1)$$

$$\Rightarrow y+2x+1=0$$

Similarly, the equation of the normal line at the point (−1,1) is

$$y-1=\frac{1}{2}(x+1)$$
$$\Rightarrow 2y-x-3=0$$

(II) Maclaurin and Taylor series

The Maclaurin series expansion for $y=f(x)$ (as a power in *x*) is defined by.

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{(4)}(0)+.....+\frac{x^n}{n!}f^{(n)}(0)+....$$

Example: Find the Maclaurin series for $f(x)=e^x$, and hence for $f(x)=\sinh x$.

Solution: We have

$$\because f(x)=e^x \quad \Rightarrow f(0)=1$$
$$\Rightarrow f'(x)=e^x \quad \Rightarrow f'(0)=1$$
$$\Rightarrow f''(x)=e^x \quad \Rightarrow f''(0)=1$$
$$\Rightarrow f'''(x)=e^x \quad \Rightarrow f'''(0)=1$$
$$\vdots$$
$$\Rightarrow f^{(n)}(x)=e^x \quad \Rightarrow f^{(n)}(0)=1$$

Now, using the Maclaurin expansion, we get

$$\therefore e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+.......$$
$$\Rightarrow e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+.......$$
$$\Rightarrow \sinh x=\frac{e^x-e^{-x}}{2}=\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}+.......$$

Example: Find the Maclaurin series for $f(x)=\sin x$ and hence for $f(x)=\cos x$.

Solution: We know that the Maclaurin series expansion for $y = f(x)$ is defined by.

Now, we have

$\because f(x) = \sin x \Rightarrow f(0) = 0$

$\Rightarrow f'(x) = \cos x \Rightarrow f'(0) = 1$

$\Rightarrow f''(x) = -\sin x \Rightarrow f''(0) = 0$

$\Rightarrow f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$\Rightarrow f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0$

$\Rightarrow f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = 1$

Hence, we have

$$\therefore \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ....$$

$$\Rightarrow \cos x = \frac{d}{dx}\left[\sin x\right] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{5!} + ...$$

Remark:

The Maclaurin expansions for $\cosh x, \cos x, \ln(1+x), \ln(1-x), \tan^{-1} x$ and more are readily found using similar application. However note that a power series expression for $\ln x$ is not possible since $\ln 0$ is not defined, and neither are derivatives of $\ln x$ at $x = 0$

Definition: The Talyor series expansion for $y = f(x)$ at $x = a$ (as a power in terms of *x*-*a*) is defined by.

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{(4)}(a)$$

$$+ ..... + \frac{(x-a)^n}{n!} f^{(n)}(a) + ...$$

OR

$$f(x+a)=f(a)+\frac{x}{1!}f'(a)+\frac{x^2}{2!}f''(a)+\frac{x^3}{3!}f'''(a)+\frac{x^4}{4!}f^{(4)}(a)+.....+\frac{x^n}{n!}f^{(n)}(a)+....$$

This means that Maclaurin series expansion is the Taylor expansion at $x = 0$.

Example: Find the expansion of $\cos(a+x)$, and hence the expansion for $\cos(x)$.

Solution: We have

$f(x)=\cos x \qquad f(a)=\cos a$

$f'(x)=-\sin x \qquad f(a)=-\sin a$

$f''(x)=-\cos x \qquad f''(a)=-\cos a$

$f'''(x)=\sin x \qquad f'''(a)=\sin a$

The pattern now established

$$\cos(a+x)=\cos a-(\sin a)x-(\cos a)\frac{x^2}{2!}+(\sin a)\frac{x^3}{3!}+....$$

Note that putting $a=0$ in this expansion

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-.....$$

Example: Find the expansion (the binomial expansion.) for $(a+x)^n$.

Solution: We have

$f(x)=x^n \qquad f(a)=a^n$

$f'(x)=nx^{n-1} \qquad f'(a)=na^{n-1}$

$f''(x)=n(n-1)x^{n-2} \qquad f''(a)=n(n-1)a^{n-2}$

$\vdots$

$$(a+x)^n=a^n+na^{n-1}+n(n-1)a^{n-2}\frac{x^2}{2!}+...$$

readily recognizable as the binomial expansion.

Remark: The series can be used for approximation. For example in the expansion for $\cos(a+x)$, aforegoing, putting $a=\frac{\pi}{3}$ (60°) and $x=\frac{\pi}{180}$ (1°) gives an approximate value for $\cos 61^\circ$. The more terms taken, the better the approximation.

So $\cos 61^\circ = \cos\frac{\pi}{3} - \left(\sin\frac{\pi}{3}\right)\left(\frac{\pi}{180}\right) - \left(\cos\frac{\pi}{3}\right)\frac{\left(\frac{\pi}{180}\right)^2}{2!} + \left(\sin\frac{\pi}{3}\right)\frac{\left(\frac{\pi}{180}\right)^3}{3!} + \ldots.$

Example: Find expansion for $\ln x$ in terms of powers of $(x-a)$.

Solution: One can be found using this first form of Taylor expansion. With

$$f(x)=\ln x \quad , \quad f(a)=\ln a$$

$$f'(x)=\frac{1}{x} \quad , \quad f'(a)=\frac{1}{a}$$

$$f''(x)=-\frac{1}{x^2} \quad , \quad f''(a)=-\frac{1}{a^2}$$

$$f'''(x)=\frac{2}{x^3} \quad , \quad f'''(a)=\frac{2}{a^3}$$

etc..

$$\therefore \ \ln x = \ln a + \frac{1}{a}(x-a) - \frac{1}{a^2}\frac{(x-a)^2}{2!} + \frac{2}{a^3}\frac{(x-a)^3}{3!} + \ldots$$

Remark:

Approximations when *x* is close to *a* such that $(x-a)$ is small, and consequently successive powers of $(x-a)$ [eg $(x-a)^2$, $(x-a)^3$] become smaller and smaller approximate values can be found

Example: In the a foregoing an approximate value of $\ln(1.01)$ can be found by letting $x=1.01$, $a=1$ and $(x-a)=0.01$

$$\ln 1.01 = \ln 1 + \frac{1}{1}(0.01) - \frac{1}{1^2}\frac{(0.01)^2}{2!} + \frac{2}{1^3}\frac{(0.01)^3}{3!} + \ldots$$

$$\therefore \quad = 0 + 0.01 - 0.00005 + 0.000000333..$$

$$= \underline{\underline{0.0099503}}$$ Compare with calculator

Other approximations:

(i) ***A linear approximation*** for $f(x)$ when <u>x is close to a</u> can be found by ignoring $(x-a)^2$ and higher powers, for then

$$f(x) \approx f(a) + (x-a)f'(a)$$

(ii) ***A quadratic approximation*** for $f(x)$ when <u>x is close to a</u> can be found by ignoring $(x-a)^3$ and higher powers, for then

$$f(x) \approx f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$$

Example: Find the linear approximation for $\tan x$ when x is close to $\frac{\pi}{4}$.

Solution:

$f(x) = \tan x \qquad f(a) = \tan a \qquad \therefore \; f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$

$f'(x) = \sec^2 x \qquad f'(a) = \sec^2 a \qquad \therefore \; f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = 2.$

$\therefore \; f(x) \approx 1 + \left(x - \frac{\pi}{4}\right)2. \qquad \underline{\underline{f(x) \approx 1 - \frac{\pi}{2} + 2x}}$

{As an exercise, find the equation of the tangent to $y = \tan x$ at $x = \frac{\pi}{4}$ and find that it is $\underline{\underline{y \approx 1 - \frac{\pi}{2} + 2x}}$ }

Example: Find the quadratic approximation for $\cos x$ when x is close to π

Solution:

$f(x) = \cos x$ $\quad f(a) = \cos a$ $\quad f(\pi) = \cos\pi = -1$

$f'(x) = -\sin x$ $\quad f'(a) = -\sin a$ $\quad f'(\pi) = -\sin\pi = 0$

$f''(x) = -\cos x$ $\quad f''(a) = -\cos a$ $\quad f''(\pi) = -\cos\pi = 1$

$$\therefore \quad f(x) \approx -1 + \frac{(x-\pi)^2}{2}$$

Further work on infinite series:

Term by term differentiation and integration can be useful.

Example: Since $\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x}$

And $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - .....$

Then $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + A$

When $x = 0$ $\quad \ln(1+x) = 0 \quad \therefore \; A = 0$

$$\therefore \; \underline{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...}$$

Example: $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + ..$

$$\therefore \quad \sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^6 + ... + A$$

when $x = 0$ $\quad \sin^{-1} x = 0 \quad \therefore \; A = 0$

$$\therefore \quad \underline{\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \frac{5}{112}x^6 + ...}$$

(III) **Critical points, Increasing and Decreasing Functions, relative maximum and relative minimum po*ints:***

The derivative of a function may be used to determine whether the function is increasing or decreasing on any intervals in its domain.

Definition: Let $y = f(x)$ be a differentiable function. Then, we have

1) A point *c is* called ***critical*** point of $f(x)$ if $f'(c) = 0$.
2) The function is said to be ***increasing*** on I if $f'(x) > 0$ at each point in an interval I.
3) The function is said to be ***decreasing*** on I if $f'(x) < 0$ at each point in an interval I.

Remark: We use the first derivative test Or the second derivative test to check that a critical point $x = c$ of a function $y = f(x)$ is relative maximum or relative minimum or neither nor
(See below)

First Derivative Test

Suppose that $x = c$ is a critical point of $f(x)$ then,

1. If $f'(x) > 0$ to the left of $x = c$ and $f'(x) < 0$ to the right of $x = c$ then $x = c$ is a relative maximum.
2. If $f'(x) < 0$ to the left of $x = c$ and $f'(x) > 0$ to the right of $x = c$ then $x = c$ is a relative minimum.
3. If $f'(x)$ is the same sign on both sides of $x = c$ then $x = c$ is neither a relative maximum nor a relative minimum.

Second Derivative Test

Suppose that $x = c$ is a critical point of $f'(c)$ such that $f'(c) = 0$ and that $f''(x)$ is continuous in a region around $x = c$. Then,

1. If $f''(c) < 0$ then $x = c$ is a relative maximum.
2. If $f''(c) > 0$ then $x = c$ is a relative minimum.
3. If $f''(c) = 0$ then $x = c$ can be a relative maximum, relative minimum or neither.

Example 1 Determine all the critical points for the function.

$$f(x) = 6x^5 + 33x^4 - 30x^3 + 100$$

~~**Solution**~~

We first need the derivative of the function in order to find the critical points and so let's get that and notice that we'll factor it as much as possible to make our life easier when we go to find the critical points.

$$\begin{aligned} f'(x) &= 30x^4 + 132x^3 - 90x^2 \\ &= 6x^2(5x^2 + 22x - 15) \\ &= 6x^2(5x - 3)(x + 5) \end{aligned}$$

Now, our derivative is a polynomial and so will exist everywhere. Therefore the only critical points will be those values of x which make the derivative zero. So, we must solve.

$$6x^2(5x - 3)(x + 5) = 0$$

Because this is the factored form of the derivative it's pretty easy to identify the three critical points. They are,

$$x = -5, \qquad x = 0, \qquad x = \frac{3}{5}$$

Example Determine all the critical points for the function.

$$h(t) = 10te^{3-t^2}$$

Solution

Here's the derivative for this function.

$$h'(t) = 10e^{3-t^2} + 10te^{3-t^2}(-2t) = 10e^{3-t^2} - 20t^2e^{3-t^2}$$

Now, this looks unpleasant, however with a little factoring we can clean things up a little as follows,

$$h'(t) = 10e^{3-t^2}(1-2t^2)$$

This function will exist everywhere and so no critical points will come from that. Determining where this is zero is easier than it looks. We know that exponentials are never zero and so the only way the derivative will be zero is if,

$$1 - 2t^2 = 0$$
$$1 = 2t^2$$
$$\frac{1}{2} = t^2$$

We will have two critical points for this function.

$$t = \pm\frac{1}{\sqrt{2}}$$

***Steps to determining increasing or decreasing intervals* and relative maximum (minimum) points**

1) Find the domain of $f(x)$ and calculate the first derivative to finding all its critical points .
2) Test all intervals in the domain of the function to the left and to the right of these critical points to determine if the derivative is positive or negative. (If $f'(x) > 0$, then f is increasing on the interval, and if $f'(x) < 0$, then f is decreasing on the interval).
3) Check that a critical point $x = c$ of a function $y = f(x)$ is relative maximum or relative minimum or neither nor using the first derivative test (or the second derivative test) .

Example: For $f(x) = x^4 - 8x^2$ determine all intervals where f is increasing or decreasing.

Solution:

The domain of $f(x)$ is all real numbers, and its derivatives given by

$$f'(x) = 4x^3 - 16x = 4x(x-2)(x+2) = 0$$

So its critical points occur at $x = -2$, 0, and 2.

Testing all intervals to the left and right of these values for $f'(x) = 4x^3 - 16x$, you find that

$$f'(x) < 0 \text{ on } (-\infty, -2)$$
$$f'(x) > 0 \text{ on } (-2, 0)$$
$$f'(x) < 0 \text{ on } (0, 2)$$
$$f'(x) > 0 \text{ on } (2, +\infty)$$

Hence, f is increasing on $(-2,0)$ and $(2,+\infty)$ and decreasing on $(-\infty, -2)$ and $(0,2)$.

Example: For $f(x) = \sin x + \cos x$ on $[0,2\pi]$, determine all intervals where f is increasing or decreasing.

Solution:

The domain of $f(x)$ is restricted to the closed interval $[0,2\pi]$, and its critical points occur at $\pi/4$ and $5\pi/4$. Testing all intervals to the left and right of these values for $f'(x) = \cos x - \sin x$, you find that

$$f'(x) > 0 \text{ on } \left[0, \frac{\pi}{4}\right)$$
$$f'(x) < 0 \text{ on } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$
$$f'(x) > 0 \text{ on } \left(\frac{5\pi}{4}, 2\pi\right]$$

Hence, f is increasing on $[0, \pi/4](5\pi/4, 2\pi)$ and decreasing on $(\pi/4, 5\pi/4)$.

Example 3 Find and classify all the critical points of the following function. Give the intervals where the function is increasing and decreasing.

$$g(t) = t\sqrt[3]{t^2 - 4}$$

Solution

First we'll need the derivative so we can get our hands on the critical points. Note as well that we'll do some simplification on the derivative to help us find the critical points.

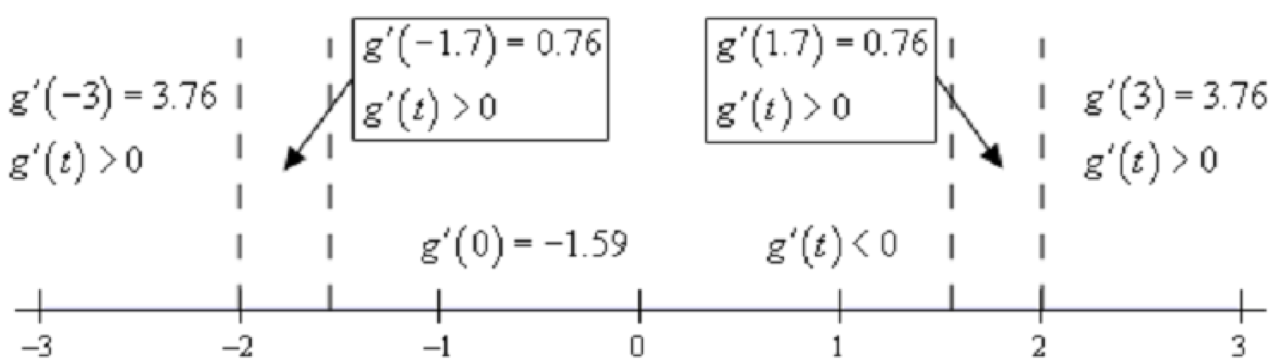
$$\begin{aligned} g'(t) &= \left(t^2-4\right)^{\frac{1}{3}} + \frac{2}{3}t^2\left(t^2-4\right)^{-\frac{2}{3}} \\ &= \left(t^2-4\right)^{\frac{1}{3}} + \frac{2t^2}{3\left(t^2-4\right)^{\frac{2}{3}}} \\ &= \frac{3\left(t^2-4\right)+2t^2}{3\left(t^2-4\right)^{\frac{2}{3}}} \\ &= \frac{5t^2-12}{3\left(t^2-4\right)^{\frac{2}{3}}} \end{aligned}$$

So, it looks like we'll have four critical points here. They are,

$t = \pm 2$ The derivative doesn't exist here.

$t = \pm\sqrt{\frac{12}{5}} = \pm 1.549$ The derivative is zero here.

Finding the intervals of increasing and decreasing will also give the classification of the critical points so let's get those first. Here is a number line with the critical points graphed and test points.



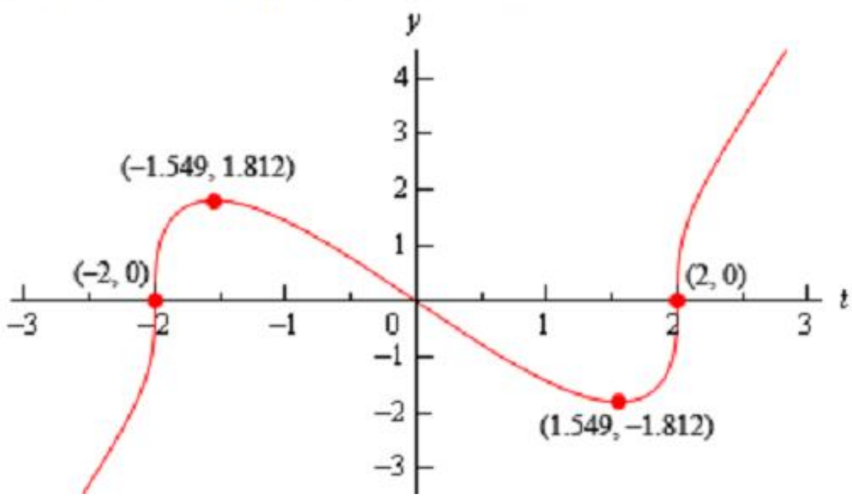
So, it looks like we've got the following intervals of increasing and decreasing.

$$\text{Increase :}\quad -\infty < x < -\sqrt{\frac{12}{5}} \text{ and } \sqrt{\frac{12}{5}} < x < \infty$$

$$\text{Decrease :}\ -\sqrt{\frac{12}{5}} < x < \sqrt{\frac{12}{5}}$$

From this it looks like $t = -2$ and $t = 2$ are neither relative minimum or relative maximums since the function is increasing on both side of them. On the other hand, $t = -\sqrt{\frac{12}{5}}$ is a relative maximum and $t = \sqrt{\frac{12}{5}}$ is a relative minimum.

For completeness sake here is the graph of the function.



Example 1 Determine all intervals where the following function is increasing or decreasing.

$$f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$$

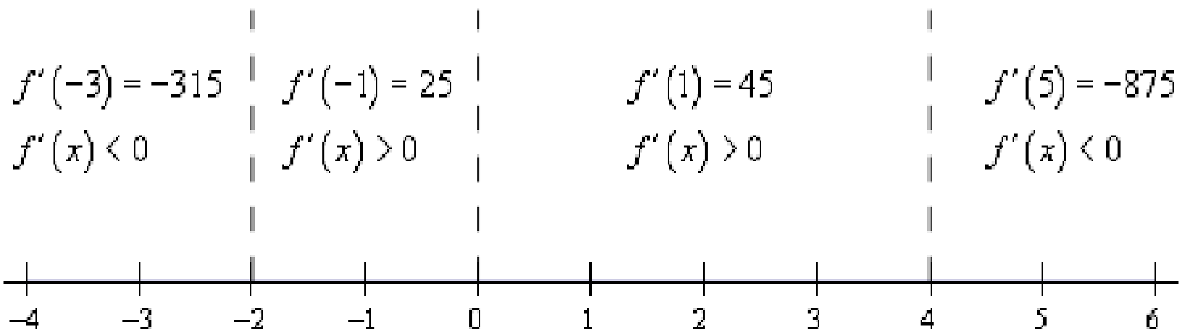
Solution

To determine if the function is increasing or decreasing we will need the derivative.

$$f'(x) = -5x^4 + 10x^3 + 40x^2$$
$$= -5x^2\left(x^2 - 2x - 8\right)$$
$$= -5x^2\left(x-4\right)\left(x+2\right)$$

From the factored form of the derivative we see that we have three critical points : $x = -2$, $x = 0$, and $x = 4$. We'll need these in a bit.

Here is the number line and the test points for the derivative.



So, it looks we've got the following intervals of increase and decrease.

$$\text{Increase}: \quad -2 < x < 0 \text{ and } 0 < x < 4$$
$$\text{Decrease}: \quad -\infty < x < -2 \text{ and } 4 < x < \infty$$

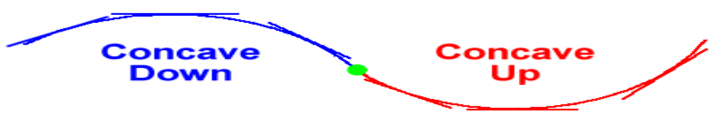
Note that often the fact that only a single point separates the two intervals of increase will be ignored and the interval will be written $-2 < x < 4$.

(IV) Concave Up, Concave Down, Points of Inflection:

We have seen previously that the sign of the derivative provides us with information about where a function (and its graph) is increasing, decreasing or stationary. We now look at the "direction of bending" of a graph, i.e. whether the graph is "concave up" or "concave down".

Definition: Let $y = f(x)$ be a differentiable function. A graph of $y = f(x)$ is said to be

1) ***concave up*** at a point if the tangent line to the graph at that point lies below the graph in the vicinity of the point (where $f''(x) > 0$)
2) ***concave down*** at a point if the tangent line lies above the graph in the vicinity of the point (where $f''(x) < 0$).



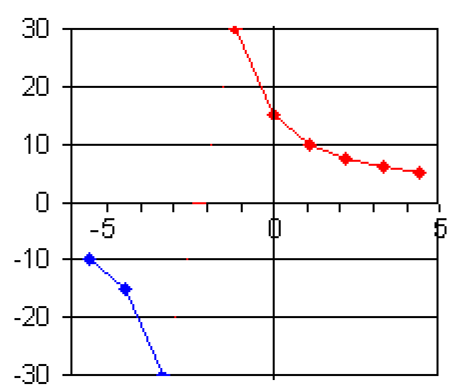
3) A point where the concavity changes (from up to down or down to up) is called a ***point of inflection*** **(POI)** (where $f''(x) = 0$). Note that the tangent line to a graph at a point of inflection must cross the graph at that point.

Steps to determining the concavity of the function's curve at any point.

1. Calculate the second derivative.
2. Substitute the value of x.
3. If $f''(x) > 0$, the graph is concave upward at that value of x.
4. If $f''(x) = 0$, the graph may have a point of inflection at that value of x. To check, consider the value of $f''(x)$ at values of x to either side of the point of interest.
5. If $f''(x) < 0$, the graph is concave downward at that value of x.

Example:

What is the shape of the graph of $f(x) = \dfrac{30}{x+2}$ at $x = 0$?



Solution:

Calculate $f''(x)$. $f'(x) = \dfrac{-30}{(x+2)^2}$ and $f''(x) = \dfrac{60}{(x+2)^3}$

When $x = 0$, $f''(x) = \frac{60}{(0+2)^3} = \frac{15}{2}$

$f''(x) > 0$, so the graph is concave upward at $x = 0$.

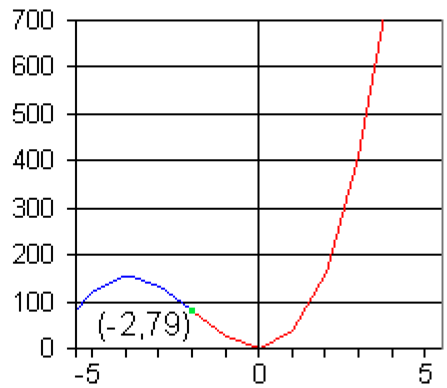
In fact $f''(x) > 0$ for all $x > -2$ so the graph is concave upward in that range. $f''(x) < 0$ for all $x < -2$ so the graph is concave downward if $x < -2$.

Similarly, we can find the points of inflection on a function's graph by calculation.

1. Calculate the second derivative.
2. Solve the equation $f''(x) = 0$ to obtain the value(s) of x at the possible point(s) of inflection. Check the $f''(x)$ value to either side of each x value to be sure each gives a point of inflection.
3. Substitute the value(s) of x into $f(x)$.
4. Hence deduce the coordinates of the point(s) of inflection.

Example: Find the point of inflection for $\boldsymbol{f(x) = 5x^3 + 30x^2 + x + 1}$.

Solution:



If $f(x) = 5x^3 + 30x^2 + x + 1$

then $f'(x) = 15x^2 + 60x + 1$

and $f''(x) = 30x + 60$.

At points of inflection, $f''(x) = 0$.

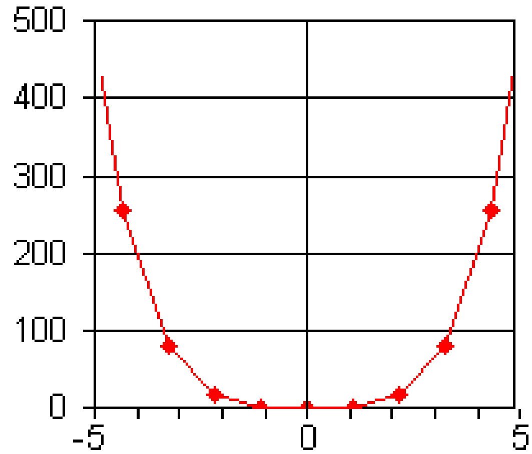
This is true when $0 = 30x + 60$, that is $x = -2$.

$f(-2) = 5(-2)^3 + 30(-2)^2 + (-2) + 1 = 79$

Below $x = -2$, the value of the second derivative, $30x + 60$, will be negative so the curve is concave down. For higher values of x, the value of the second derivative, $30x + 60$, will be positive so the curve is concave up. We can conclude that the point (-2,79) is a point of inflection.

Example: Find the point of inflection for $\boldsymbol{f(x) = x^4}$.

Solution:
Consider $f(x) = x^4$. Solving $f''(x) = 12x^2 = 0$ yields $x = 0$. At values of $x < 0$, the second derivative is positive. At values of $x > 0$, the second derivative is positive. (0,0) is a local minimum (is not inflection).



(V) Indeterminate Forms and L'Hospital's Rule:

What happens when we try to evaluate $\lim_{x\to 1}\frac{\ln x}{x-1}$?

a) Can't do the limits of the numerator and denominator separately. Why?
b) No common terms to cancel

The limit may or may not exist and is called an **indeterminate form of type** ∞/∞

There is also an indeterminate form of type $0/0$. Ex: $\lim_{x\to 0}\frac{x}{x^2}$

L'Hospital's Rule

Suppose that we have one of the following cases,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0} \qquad \text{OR} \qquad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example: Evaluate the following limits

$i)\quad \underset{x\to 0}{Lim}\frac{\sin 5x}{3x}$, $\qquad ii)\quad \underset{x\to 0}{Lim}\frac{\ln(1+x)}{x}$

Solution:

$i)\quad \because \underset{x\to 0}{Lim}\frac{\sin 5x}{3x}=\frac{0}{0}$

Hence, using L'Hospital's rule, we get

$$\therefore \underset{x\to 0}{Lim}\frac{\sin 5x}{3x}=\underset{x\to 0}{Lim}\frac{5\cos 5x}{3}=\frac{5}{3}$$

$ii)\quad \because \underset{x\to 0}{Lim}\frac{\ln(1+x)}{x}=\frac{0}{0}$

Hence, using L'Hospital's rule, we obtain

$$\therefore \underset{x\to 0}{Lim}\frac{\ln(1+x)}{x}=\underset{x\to 0}{Lim}\frac{1}{1+x}=1$$

Example: Evaluate the following limits

$i)\ \underset{x\to 0}{Lim}\frac{e^{x}-e^{-x}-2x}{x-\sin x}$

$ii)\ \underset{x\to 0}{Lim}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$

$iii)\ \underset{x\to 0}{Lim}(\cos ec\, x-\cot x)$

Solution: Using *L'Hospital's Rule* for three times, we get

$$i)\ \lim_{x\to 0}\frac{e^{x}-e^{-x}-2x}{x-\sin x}=\lim_{x\to 0}\frac{e^{x}+e^{-x}-2}{1-\cos x}=\lim_{x\to 0}\frac{e^{x}-e^{-x}}{\sin x}$$

$$-\therefore \lim_{x\to 0}\frac{e^{x}-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{e^{x}+e^{-x}}{\cos x}=\frac{2}{1}=2$$

$$ii)\ \lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}=\frac{1-1}{0}=\frac{0}{0}$$

Hence, using L'Hospital's rule, we get

$$ii)\ \lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}=\frac{0}{0}$$

Hence, using L'Hospital's rule, we get

$$\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}=\lim_{x\to 0}\frac{\frac{\cos x}{2\sqrt{1+\sin x}}+\frac{\cos x}{2\sqrt{1-\sin x}}}{1}$$

$$=\frac{1}{2}+\frac{1}{2}=1$$

$$iii)\ \lim_{x\to 0}(\operatorname{cosec} x-\cot x)=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)=\frac{1-1}{0}=\frac{0}{0}$$

$$=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)=\lim_{x\to 0}\left(\frac{\sin x}{\cos x}\right)=0$$

L'Hospital's Rule works great on the two indeterminate forms 0/0 and $\pm\infty\pm\infty$.
However, there
are many more indeterminate forms { $0\cdot\mp\infty, \infty-\infty, 0^{0}, \infty^{0}, 1^{\infty}$ }. Let's take a look at some of
those and see how we deal with those kinds of indeterminate forms.

We'll start with the indeterminate form $(0\cdot\mp\infty)$.

*What about **indeterminate products***? $(0 \cdot \mp\infty)$ ➔Turn them into quotients!

Example: Evaluate the following limit

$$\lim_{x\to 0^+}(x\ln x)$$

Solution: Since, $\lim_{x\to 0^+}(x\ \ln x) = 0\cdot(-\infty)$

Now, in the limit, we get the indeterminate form $(0)(-\infty)$. L'Hospital's Rule won't work on products, it only works on quotients. However, we can turn this into a fraction if we rewrite things a little.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x}$$

The function is the same, just rewritten, and the limit is now in the form $-\infty/\infty$ and we can now use L'Hospital's Rule.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2}$$

Now, this is a mess, but it cleans up nicely.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

Same idea for differences (∞ − ∞)…

Example: Evaluate the following limit

$$\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$$

Solution:

(see the preceding Example (iii))

Indeterminate powers: $0^0, \infty^0, 1^\infty$

a) take the natural logarithm
b) write the function as an exponential

Example: Evaluate the following limit

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$$

Solution: Since $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = 1^{\infty}$ (indeterminate power), putting

$$y = (1+\sin 4x)^{\cot x} \quad \Rightarrow \lim_{x\to 0^+} y \ ?$$

Taking the natural logarthim ln, we have

$$\ln y = \cot \ln(1+\sin 4x) = \frac{\ln(1+\sin 4x)}{\tan x}$$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} = \frac{0}{0}$$

Now, we can use L'Hospital rules

$$\ln \lim_{x\to 0^+} y = \lim_{x\to 0^+} \frac{4\cos 4x \big/ (1+\sin 4x)}{\sec^2 x}$$

$$= \lim_{x\to 0^+} \frac{4\cos 4x}{\sec^2 x\,(1+\sin 4x)}$$

$$= \lim_{x\to 0^+} \frac{4\cos 4x \cos^2 x}{(1+\sin 4x)} = 1$$

Hence, we obtain

$$\lim_{x\to 0^+} y = e$$

EXERCISES

1) Find the equation of the tangent and normal lines to following curves at the corresponding points:

$i)\ x^2 + y^2 - 4x + 6y - 3 = 0 \quad ; \quad (2.-7)$

$ii)\ x^2 - y^2 = 9 \quad ; \quad (3\sqrt{2},3)$

$iii)\ y^2 = x - 1 \quad ; \quad (5,2)$

$iv)\ x = 2y^2 - y + 1 \quad ; \quad (4,-1)$

$v)\ x = \ln t \quad ; \quad y = 2t^2 \quad ; \quad (0,2)$

2) Find the equation of the tangent $y = x^3 - 6x + 2$ and parallel to the line $y = 6x - 2$.

3) Prove the two tangent lines for $x^2 - 4y + 4 = 0$ at $(3/2,0)$ are perpendicular.

4) Determine the increasing and decreasing intervals for the following functions:

$b)\ y = 2x^2 - 4x + 5 \qquad a)\ y = x^2 - 4x + 1$

$d)\ y = x^3 - 3x^2 + 5 \qquad c)\ y = 2x^3 - 3x^2 + 1$

$e)\ y = x^4 - 2x^2 + 1 \qquad f)\ y = x^4 - 4x^2 + 5$

5) Find the maximum and minimum extremes for the following functions:

$a)\ y = x^2 - 2x + 3 \qquad b)\ y = x^4 - 8x^2 + 2$

$c)\ y = -x^4 + 2x^3 \qquad d)\ y = x^3 - 9x^2 + 15x + 3$

6) Evaluate the following limits;

$i)\ \underset{x\to 0}{Lim}\ \frac{1-\cos x}{x^2} \qquad ii)\ \underset{x\to \pi/2}{Lim}\ \frac{2\cos x}{2x-\pi}$

$iii)\ \underset{x\to 0}{Lim}\ \frac{\cos^{-1}(1+x)}{\ln(1+x)} \qquad iv)\ \underset{x\to 0}{Lim}\ \frac{x-\tan x}{x-\sin x}$

$(v)\ \lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} \qquad (vi)\ \lim_{x\to 0} \frac{\tan x - x}{x^3}$

7) Determine the point(s) on $y = x^2 + 1$ that are closest to (0,2).

8) Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.
9) For the following function find the inflection points and use the second derivative test, if possible, to classify the critical points. Also, determine the intervals of increase , decrease and the intervals of concave up, concave down and sketch the graph of the functions: $(a)\quad y = x(6-x)^{\frac{2}{3}}$

   $(b)\; y = x^3 + 3x^2 - 9x - 20$
10) Use the second derivative test to classify the critical points of the function $y = 3x^5 - 5x^3 + 3$

Part II
Integration and its applications